

Equivalents of the Axiom of Choice

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Abstract

This document shows the equivalence of several important mathematical principles, starting with the Axiom of Choice and going through a circular proof path back to the Axiom of Choice. The main statements covered include Zermelo's Well-Ordering Theorem, Transfinite Recursion, Hausdorff's Maximal Principle, Order-Extension Principle, Existence of Basis for Vector Spaces, Tychonoff's Theorem, and Zorn's Lemma.

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1 Introduction

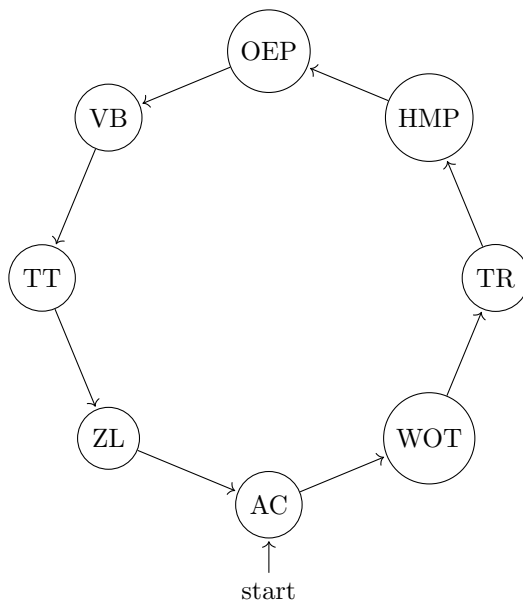
The Axiom of Choice (AC) is one of the most important, yet controversial, principles in the field of set theory and mathematical logic. This axiom states that given any collection of non-empty sets, it is possible to select exactly one element from each set, even if the collection is infinite. Despite its apparent simplicity, the Axiom of Choice has far-reaching implications across various branches of mathematics.

Several mathematical statements have been shown to be equivalent to the Axiom of Choice, meaning that if the Axiom of Choice is assumed to be true, these statements are also true, and vice versa. This document explores these equivalent statements, illustrating how one can be derived from another in a circular chain of implications that ultimately returns to the Axiom of Choice.

The other main statements covered in this document include:

- **Zermelo's Well-Ordering Theorem (WOT):** Every set can be well-ordered.
- **Transfinite Recursion (TR):** Given a well-ordered set, any function defined on all smaller elements can be used to define an element of the set.

Figure 1: A circle of implications between equivalent principles of the Axiom of Choice



AC	-	Axiom of Choice
WOT	-	Zermelo's Well-Ordering Theorem
TR	-	Transfinite Recursion
HMP	-	Hausdorff's Maximal Principle
OEP	-	Order-Extension Principle
VB	-	Existence of Basis for Vector Spaces
TT	-	Tychonoff's Theorem
ZL	-	Zorn's Lemma

- **Hausdorff's Maximal Principle (HMP):** In any partially ordered set, every chain is contained in a maximal chain.
- **Order-Extension Principle (OEP):** Any partial order can be extended to a total order.
- **Existence of Basis for Vector Spaces (VB):** Every vector space has a basis.
- **Tychonoff's Theorem (TT):** The product of any collection of compact topological spaces is compact in the product topology.
- **Zorn's Lemma (ZL):** Every partially ordered set in which every chain has an upper bound contains at least one maximal element.

The relationships between these principles are illustrated in Figure 1, where each arrow represents a proof of implication from one principle to another. By following these implications, we can see the interconnectedness of these fundamental concepts and their equivalence to the Axiom of Choice.

2 Axiom of Choice Implies the Well-Ordering Theorem

2.1 Axiom of Choice

The Axiom of Choice (AC) is a fundamental principle in set theory and can be formally stated as follows:

$$\forall \mathcal{A} (\mathcal{A} \neq \emptyset \wedge \forall X \in \mathcal{A}, X \neq \emptyset \implies \exists f : \mathcal{A} \rightarrow \bigcup \mathcal{A} (\forall X \in \mathcal{A}, f(X) \in X)).$$

This means:

- $\mathcal{A}$ is a set of non-empty sets.
- There exists a function f such that f maps every set $X \in \mathcal{A}$ to an element within X . Formally, $f(X) \in X$.

Intuitively, the Axiom of Choice allows us to make an arbitrary selection of elements from each set in a collection of non-empty sets, even if the collection is infinite.

2.2 Well-Ordering Theorem

The Well-Ordering Theorem (WOT) states:

$$\forall S, \exists R, \exists x \in S \text{ s.t. } (\forall y \in S, x \leq y)$$

meaning:

- S is any set.
- There exists a total order R on S such that, for every subset, there is an element within that subset that is less than or equal to all other elements within it, i.e every non-empty subset of S has a least element.

In essence, the theorem guarantees the existence of an order relationship for any set such that within this order, every subset has a first element.

2.3 Proof of Implication

1. **Given:** The Axiom of Choice (AC)
2. **To Prove:** The Well-Ordering Theorem (WOT)

3. Proof:

Let S be a set. We need to well-order S . By AC, there exists a choice function f for every non-empty subset of S .

4. Define the ordering:

- (a) Let $\mathcal{P}(S)$ be the power set of S , i.e., the set of all subsets of S .
- (b) Define a function $g : \mathcal{P}(S) \rightarrow S$ such that for any non-empty subset $A \subseteq S$, $g(A) = f(A)$ where f is the choice function guaranteed by AC.

5. Construct the well-order:

- (a) We construct the well-ordering using the following transfinite recursion:
 - Let $A_0 = S$ and define $a_0 = g(A_0)$.
 - The choice function g chooses an element $a_0 \in S$.
 - Define $A_1 = A_0 \setminus \{a_0\}$ and pick $a_1 = g(A_1)$.
 - Continue this process: if $A_\alpha \subseteq S$ has been defined for an ordinal α , and $A_\alpha \neq \emptyset$, then pick $a_\alpha = g(A_\alpha)$, and let $A_{\alpha+1} = A_\alpha \setminus \{a_\alpha\}$.
 - If β is a limit ordinal, define $A_\beta = S \setminus \{a_\alpha : \alpha < \beta\}$.
- (b) Repeat this process until $A_\gamma = \emptyset$ for some ordinal γ . Since S is a set and ordinals are well-ordered, such a γ must exist (S is exhausted).

6. Define the order relation:

- (a) Define a relation R on S by declaring $a_\alpha \leq a_\beta \iff \alpha \leq \beta$. Here α and β are ordinals in the transfinite sequence constructed.
- (b) This relation R is a total order on S :
 - It is antisymmetric: If $a_\alpha \leq a_\beta$ and $a_\beta \leq a_\alpha$, then $\alpha = \beta$.
 - It is transitive: If $a_\alpha \leq a_\beta$ and $a_\beta \leq a_\gamma$, then $a_\alpha \leq a_\gamma$ because $\alpha \leq \beta$ and $\beta \leq \gamma$ imply $\alpha \leq \gamma$.
 - It is total: For any $a_\alpha, a_\beta \in S$, either $\alpha \leq \beta$ or $\beta \leq \alpha$.
- (c) To show it is a well-order, observe that any non-empty subset $S' \subseteq S$ has a least element: The process of choice using g ensures that the element chosen first from S' will be its least element.

$\therefore$ Therefore, the Axiom of Choice implies the Well-Ordering Theorem.

□

2.4 Well-Ordering Theorem Implies Transfinite Recursion

2.4.1 Well-Ordering Theorem

Recall that the Well-Ordering Theorem (WOT) states that every set S can be well-ordered, i.e., there exists a total order R on S such that every non-empty subset of S has a least element.

2.4.2 Transfinite Recursion

Transfinite Recursion (TR) states:

For any well-ordered set $(W, \leq)$ and a function G defined on all subsets of W prior to x ,

there exists a unique function $f : W \rightarrow V$ such that for all $x \in W$, $f(x) = G(f|_{<x})$.

Explicitly:

- W is a well-ordered set.
- G is a function that assigns values based on functions defined on initial segments of W .
- There exists a function f that is defined in a manner consistent with the rule given by G .

This principle allows function definitions using sets of ordinals, facilitating constructions across an entire order by defining elements based on previously defined ones. This principle is particularly useful in set theory and ordinal analysis.

2.4.3 Proof of Implication

1. **Given:** The Well-Ordering Theorem (WOT)

2. **To Prove:** Transfinite Recursion (TR)

3. **Proof:**

Let $(W, \leq)$ be a well-ordered set, and let G be a function that assigns to each $f|_{<x}$ a value in some set V , where $f|_{<x}$ is the restriction of f to elements less than x .

4. **Define the function f using transfinite recursion:**

(a) Construct f by recursion on the well-order $(W, \leq)$:

- i. For each element $x \in W$, we need to assign $f(x)$ based on the values of f at all previous elements under the order $\leq$.

(b) **Basis step:**

- Let x_0 be the least element in W .
- Assign $f(x_0)$ using $G(\emptyset)$, as there are no elements less than x_0 .

(c) **Induction step:**

- Assume f has been defined for all elements less than some element $x \in W$.
- Define $f(x) = G(f|_{<x})$, where $f|_{<x}$ is the function restricted to all elements less than x .

(d) **Limit step:**

- Let γ be a limit ordinal. Assume f has been defined for all $y < \gamma$.
- Define $f(\gamma) = G(f|_{<\gamma})$, where $f|_{<\gamma}$ involves all elements less than γ .

5. **Uniqueness:**

- (a) The uniqueness of f follows from the fact that G assigns a unique value for each $f|_{<x}$.
- (b) By well-ordering, every element x uniquely determines $f(x)$ based on previously defined values.

6. **Result:**

- (a) We have constructed a unique function $f : W \rightarrow V$ that satisfies $f(x) = G(f|_{<x})$ for all $x \in W$.

$\therefore$ The Well-Ordering Theorem implies Transfinite Recursion.

□

3 Transfinite Recursion Implies Hausdorff's Maximal Principle

3.1 Transfinite Recursion

Recall that Transfinite Recursion (TR) states that for any well-ordered set $(W, \leq)$ and a function G defined on all subsets of W prior to x , there exists a unique function $f : W \rightarrow V$ such that for all $x \in W$, $f(x) = G(f|_{<x})$.

3.2 Hausdorff's Maximal Principle

Hausdorff's Maximal Principle (HMP) asserts that in any partially ordered set $(P, \leq)$, every chain (i.e., totally ordered subset) is contained in a maximal chain. Specifically:

$\forall P$ (if $\leq$ is a partial order on P , then every chain in P is contained in a maximal chain).

A chain is a subset of a partially ordered set in which every two elements are comparable. A maximal chain is a chain that cannot be properly extended by including any additional elements from the partially ordered set while maintaining the property of being a chain.

3.3 Proof of Implication

1. **Given:** Transfinite Recursion (TR)
2. **To Prove:** Hausdorff's Maximal Principle (HMP)
3. **Proof:**

Let $(P, \leq)$ be a partially ordered set, and suppose C is a chain in P . We aim to show that C is contained in a maximal chain using Transfinite Recursion.

4. **Construct a maximal chain containing C :**

- (a) Define an initial well-ordered set W and a function G that will help us build larger and larger chains.
- (b) **Define W :**
 - Let W be the set of ordinals (i.e., $W = \text{Ord}$).
- (c) **Define G :**
 - For a given function f defined on an initial segment of W , $f|_{<\alpha}$, denote by $\bigcup_{\beta < \alpha} D_\beta$ the union of all the chains described by the function at points $\beta < \alpha$.
 - Define G so that $G(f|_{<\alpha})$ assigns the next element in the chain to incorporate more possible elements from P . Specifically, let $f(\alpha)$ be an extension of the chain if possible; otherwise, let it be $\bigcup_{\beta < \alpha} D_\beta$.

5. **Use Transfinite Recursion:**

- (a) By TR, there exists a unique function $f : W \rightarrow \mathcal{P}(P)$ (the power set of P) such that $f(\alpha) = G(f|_{<\alpha})$ for all ordinals α .

6. **Define a chain D_α at each step α :**

- (a) Let $D_0 = C$ be the initial chain.
- (b) For successor ordinals $\alpha + 1$, if possible, extend D_α by adding a new element from P that maintains the chain (totally ordered), otherwise, set $D_{\alpha+1} = D_\alpha$.
- (c) For limit ordinals λ , define $D_\lambda = \bigcup_{\beta < \lambda} D_\beta$.

7. **Maximal chain:**

- (a) The construction ensures that at each stage α , D_α is a chain and $D_\alpha \subseteq D_{\alpha+1}$.
- (b) If the process terminates at some step γ , then D_γ is a maximal chain.
- (c) Otherwise, continuing the recursion generates larger and larger chains. By the nature of the ordinals and transfinite recursion, this process leads to a maximal chain in P .

8. Result:

- (a) The constructed $\bigcup_{\alpha} D_{\alpha}$ represents a maximal chain containing C , fulfilling Hausdorff's Maximal Principle.

$\therefore$ Transfinite Recursion implies Hausdorff's Maximal Principle.

□

4 Hausdorff's Maximal Principle Implies the Order-Extension Principle

4.1 Hausdorff's Maximal Principle

Hausdorff's Maximal Principle (HMP) asserts that in any partially ordered set $(P, \leq)$, every chain (i.e., totally ordered subset) is contained in a maximal chain. Specifically:

$\forall P$ (if $\leq$ is a partial order on P , then every chain in P is contained in a maximal chain).

4.2 Order-Extension Principle

The Order-Extension Principle (OEP) states that any partial order $(P, \leq)$ on a set P can be extended to a total order $(P, \leq^*)$ on the same set. Formally:

$\forall P$ (if $\leq$ is a partial order on P , then there exists a total order $\leq^*$ on P such that $\leq \subseteq \leq^*$).

A partial order is a binary relation that is reflexive, antisymmetric, and transitive, but not necessarily comparable for every pair of elements. Extending this to a total order implies that every pair of elements in the set is comparable under $\leq^*$, creating a linear arrangement of the set.

4.3 Proof of Implication

1. **Given:** Hausdorff's Maximal Principle (HMP)
2. **To Prove:** Order-Extension Principle (OEP)
3. **Proof:**

Let $(P, \leq)$ be a partially ordered set. We need to show that there exists a total order $(P, \leq^*)$ that extends $(P, \leq)$.

4. **Construct the set of partial orders:**

- (a) Consider the set $\mathcal{T}$ of all partial orders on P that extend $\leq$. Formally, $\mathcal{T} = \{\leq' \mid \leq' \text{ is a partial order on } P \text{ and } \leq \subseteq \leq'\}$.

- (b) Partially order $\mathcal{T}$ by inclusion: $\leq_1 \subseteq \leq_2$ if whenever $x \leq_1 y$, then $x \leq_2 y$ for all $x, y \in P$.

5. **Use HMP to find a maximal element:**

- (a) By HMP, every chain in $\mathcal{T}$ (with the inclusion order) is contained in a maximal element of $\mathcal{T}$.
(b) Thus, there exists a maximal partial order $\leq^* \in \mathcal{T}$ that extends $\leq$.

6. **Show $\leq^*$ is a total order:**

- (a) Assume $\leq^*$ is not a total order.
(b) Then, there exist elements $x, y \in P$ such that $x \not\leq^* y$ and $y \not\leq^* x$.
(c) Define a new relation $\leq^{**}$ on P by extending $\leq^*$ to include (x, y) or (y, x) while maintaining antisymmetry and transitivity.
(d) Since $\leq^*$ is maximal, any proper extension that forms a partial order must include either $x \leq^* y$ or $y \leq^* x$.
(e) Thus, $\leq^*$ actually accounts for all pairs (x, y) , making $\leq^*$ a total order.

$\therefore$ Hausdorff's Maximal Principle implies the Order-Extension Principle.

□

5 Order-Extension Principle Implies the Existence of Basis for Vector Spaces

5.1 Order-Extension Principle

The Order-Extension Principle (OEP) states that any partial order $(P, \leq)$ on a set P can be extended to a total order $(P, \leq^*)$ on the same set. Formally:

$\forall P$ (if $\leq$ is a partial order on P , then there exists a total order $\leq^*$ on P such that $\leq \subseteq \leq^*$).

5.2 Existence of Basis for Vector Spaces

The statement that every vector space V over a field F has a basis (VB) asserts that:

$$\forall V (\exists \mathcal{B} \subseteq V \text{ s.t. } \mathcal{B} \text{ is a linearly independent set and spans } V).$$

A basis $\mathcal{B}$ of a vector space V is a set of vectors in V that are linearly independent and such that every vector in V can be written as a linear combination of the basis vectors. This guarantees that any vector space, regardless of its dimensionality, has a well-defined basis, allowing for the unique representation of vectors.

5.3 Proof of Implication

1. **Given:** Order-Extension Principle (OEP)

2. **To Prove:** Existence of Basis for Vector Spaces (VB)

3. **Proof:**

Let V be a vector space over a field F . We need to show that there exists a subset $\mathcal{B} \subseteq V$ that is a basis for V .

4. **Construct the set of linearly independent subsets:**

- (a) Consider the set $\mathcal{I}$ of all linearly independent subsets of V .
- (b) Partially order $\mathcal{I}$ by inclusion: For $A, B \in \mathcal{I}$, define $A \leq B$ if $A \subseteq B$.

5. **Use OEP to find a maximal element:**

- (a) By OEP, there exists a total order $\leq^*$ on $\mathcal{I}$ that extends $\leq$ (inclusion).
- (b) With this total order, every chain (total ordered subset) in $\mathcal{I}$ has an upper bound in $\mathcal{I}$, i.e., a maximal element of the chain.
- (c) Therefore, we need to demonstrate that this order leads to a maximal linearly independent subset.

6. **Maximal Linearly Independent Subset is a Basis:**

- (a) Let $\mathcal{B} \subseteq V$ be a maximal linearly independent subset of V derived from Zorn's Lemma (since every chain in $\mathcal{I}$ has an upper bound).
- (b) To show $\mathcal{B}$ is a basis, we need to prove $\mathcal{B}$ spans V .
- (c) Suppose $\mathcal{B}$ does not span V . Then, there exists a vector $v \in V$ such that $v \notin \text{span}(\mathcal{B})$.
- (d) Adding v to $\mathcal{B}$, the set $\mathcal{B} \cup \{v\}$ must be linearly independent (otherwise, v could be written as a linear combination of elements of $\mathcal{B}$, contradicting $v \notin \text{span}(\mathcal{B})$).
- (e) This contradicts the maximality of $\mathcal{B}$, as $\mathcal{B} \cup \{v\}$ is a strictly larger linearly independent subset.
- (f) Therefore, $\mathcal{B}$ must span V .

$\therefore$ The maximal linearly independent subset $\mathcal{B}$ is a basis for V .

□

6 Existence of Basis for Vector Spaces Implies Tychonoff's Theorem

6.1 Existence of Basis for Vector Spaces

Recall that the statement that every vector space V over a field F has a basis (VB) asserts:

$$\forall V (\exists \mathcal{B} \subseteq V \text{ s.t. } \mathcal{B} \text{ is a linearly independent set and spans } V).$$

6.2 Tychonoff's Theorem

Tychonoff's Theorem (TT) states that the product of any collection of compact topological spaces is compact in the product topology. Formally:

If $\{X_i\}_{i \in I}$ is a collection of compact spaces, then $\prod_{i \in I} X_i$ is compact.

In more detail, a topological space X is called compact if every open cover of X has a finite subcover. Specifically, if $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets such that $X \subseteq \bigcup_{\alpha \in A} U_\alpha$, then there is a finite subset $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ such that $X \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. The product topology on $\prod_{i \in I} X_i$ is the coarsest topology for which all the projection maps $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$ are continuous.

The product topology on $\prod_{i \in I} X_i$ is the coarsest topology for which all the projection maps $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$ are continuous.

6.3 Proof of Implication

1. **Given:** Existence of Basis for Vector Spaces (VB)

2. **To Prove:** Tychonoff's Theorem (TT)

3. **Proof:**

Let $\{X_i\}_{i \in I}$ be a collection of compact topological spaces. We need to show that the product space $X = \prod_{i \in I} X_i$ is compact in the product topology.

4. **Construct the product space:**

- (a) Consider $X = \prod_{i \in I} X_i$ with the product topology.
- (b) By definition, the product topology is the coarsest topology in which all projection maps $\pi_i : X \rightarrow X_i$ are continuous.

5. **Use the Finite Intersection Property:**

- (a) To prove compactness, we use the Finite Intersection Property (FIP): a space is compact if and only if every collection of closed sets with the FIP has a non-empty intersection.

- (b) Let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed sets in X such that every finite subcollection $\{F_{\alpha_k}\}_{k=1}^n$ has a non-empty intersection.

6. Find a Vector Space Approach:

- (a) Construct V , the vector space of all functions $f : I \rightarrow \bigcup_{i \in I} X_i$ where $f(i) \in X_i$ for each $i \in I$.
- (b) For each $\alpha \in A$, define V_α to be the subspace of V consisting of functions whose inverse images of X_i -closed sets are also closed.

7. Select a Basis and Show Non-emptiness:

- (a) By VB, V has a basis $\mathcal{B}$. Since the intersections $\bigcap_{k=1}^n F_{\alpha_k} \neq \emptyset$ for any finite collection, every choice vector from $\mathcal{B}$ must meet each F_α .
- (b) Construct a function $f : I \rightarrow \bigcup_{i \in I} X_i$ using elements from $\mathcal{B}$, ensuring consistency with finite intersections.
- (c) Each finite intersection remains non-empty under the induced subspace, ensuring the entire collection's intersection stays non-empty in $\prod_{i \in I} X_i$.

$\therefore$ The product space $\prod_{i \in I} X_i$ is compact, establishing Tychonoff's Theorem.

□

7 Tychonoff's Theorem Implies Zorn's Lemma

7.1 Tychonoff's Theorem

Recall that Tychonoff's Theorem (TT) states that the product of any collection of compact topological spaces is compact in the product topology. Formally:

If $\{X_i\}_{i \in I}$ is a collection of compact spaces, then $\prod_{i \in I} X_i$ is compact.

7.2 Zorn's Lemma

Zorn's Lemma (ZL) is another important principle in set theory. It states:

Every partially ordered set $(P, \leq)$ in which every chain has an upper bound contains at least one maximal element. Essentially, Zorn's Lemma (ZL) states:

$$\forall (P, \leq), (\forall C \subseteq P, (\exists u \in P \text{ s.t. } (\forall c \in C, c \leq u)) \implies \exists m \in P \text{ s.t. } (\forall p \in P, m \geq p)).$$

In more detail:

- $(P, \leq)$ is a partially ordered set (poset).
- A chain $C \subseteq P$ is a subset of P where every pair of elements is comparable (i.e., for any $a, b \in C$, either $a \leq b$ or $b \leq a$).
- An upper bound of a chain C is an element $u \in P$ such that $\forall c \in C, c \leq u$.
- A maximal element in P is an element $m \in P$ such that there is no element $p \in P$ with $m < p$.

7.3 Proof of Implication

1. **Given:** Tychonoff's Theorem (TT)

2. **To Prove:** Zorn's Lemma (ZL)

3. **Proof:**

Let $(P, \leq)$ be a partially ordered set. Suppose every chain in P has an upper bound. We need to show that P has a maximal element.

4. **Construct the space of filters:**

- Define the set Q to be the set of all filters on P .
- Recall that a filter $\mathcal{F}$ on P is a non-empty subset of P such that:
 - For all $x, y \in \mathcal{F}$, there exists $z \in \mathcal{F}$ such that $z \leq x$ and $z \leq y$.
 - If $x \in \mathcal{F}$ and $x \leq y$, then $y \in \mathcal{F}$.

5. **Define the product topology:**

- Consider P as a discrete space.
- Let $X = \{0, 1\}^P$ be the product of P , where $\{0, 1\}$ is the discrete space.
- By Tychonoff's Theorem, X is compact.

6. **Associate filters with points in the product space:**

- For any filter $\mathcal{F} \subseteq P$, define a point $x_{\mathcal{F}} \in X$ such that:

$$x_{\mathcal{F}}(p) = \begin{cases} 1 & \text{if } p \in \mathcal{F}, \\ 0 & \text{if } p \notin \mathcal{F}. \end{cases}$$

- The set of all such points $x_{\mathcal{F}}$ corresponds to the set of filters on P .

7. **Consider a closed subset:**

- Let C be the closed subset of points $x \in X$ such that for any $a, b \in P$, if $x(a) = 1$ and $x(b) = 1$, there exists $c \in P$ such that $x(c) = 1$ and $c \leq a$ and $c \leq b$.

8. Finite Intersection Property:

- (a) Consider the family of closed sets in X that corresponds to the chains in P .
- (b) Every chain $\{F_i\}$ forms an upper bound and thus meets the finite intersection property.
- (c) By the compactness of X (from Tychonoff's Theorem), the intersection of all these closed sets is non-empty.
- (d) Hence, there exists a maximal filter, $\mathcal{M}$, in P .

9. Maximal filter corresponds to a maximal element:

- (a) Consider the maximal filter $\mathcal{M}$.
- (b) Assume $\mathcal{M}$ is not corresponding to a maximal element in P , then it can be extended: contradicting maximality.
- (c) Hence, the existence of $\mathcal{M}$ assures the existence of a maximal element in P .

$\therefore$ Tychonoff's Theorem implies Zorn's Lemma.

□

8 Zorn's Lemma Implies Axiom of Choice

8.1 Zorn's Lemma

Recall that Zorn's Lemma (ZL) states:

$$\forall(P, \leq), (\forall C \subseteq P, (\exists u \in P \text{ s.t. } (\forall c \in C, c \leq u)) \implies \exists m \in P \text{ s.t. } (\forall p \in P, m \geq p \implies m = p)).$$

In other words, every partially ordered set $(P, \leq)$ in which every chain has an upper bound contains at least one maximal element.

8.2 Axiom of Choice

The Axiom of Choice (AC) states:

$$\forall \mathcal{A} (\mathcal{A} \neq \emptyset \wedge \forall X \in \mathcal{A}, X \neq \emptyset \implies \exists f : \mathcal{A} \rightarrow \bigcup \mathcal{A} (\forall X \in \mathcal{A}, f(X) \in X)).$$

This means, given any set $\mathcal{A}$ of non-empty sets, there exists a choice function f such that $f(X) \in X$ for every $X \in \mathcal{A}$.

8.3 Proof of Implication

1. **Given:** Zorn's Lemma (ZL)
2. **To Prove:** The Axiom of Choice (AC)
3. **Proof:**

Let $\mathcal{A}$ be a set of non-empty sets. We need to construct a choice function $f : \mathcal{A} \rightarrow \bigcup \mathcal{A}$ such that $f(X) \in X$ for every $X \in \mathcal{A}$.

4. **Construct the set of partial choice functions:**

(a) Define P to be the set of all partial choice functions on $\mathcal{A}$. Formally:

$$P = \{g \mid g : D \rightarrow \bigcup \mathcal{A}, D \subseteq \mathcal{A} \text{ and } \forall X \in D, g(X) \in X\}.$$

(b) Partially order P by inclusion of functions: For $g_1, g_2 \in P$,

$$g_1 \leq g_2 \iff \text{dom}(g_1) \subseteq \text{dom}(g_2) \text{ and } \forall X \in \text{dom}(g_1), g_1(X) = g_2(X).$$

5. **Use ZL to find a maximal element:**

- (a) To apply Zorn's Lemma, we need to show that every chain (totally ordered subset) in P has an upper bound in P .
- (b) Let $\{g_\alpha\}_{\alpha \in I}$ be a chain in P . Define a function h such that:

$$\text{dom}(h) = \bigcup_{\alpha \in I} \text{dom}(g_\alpha),$$

$$h(X) = g_\alpha(X) \text{ for any } \alpha \text{ such that } X \in \text{dom}(g_\alpha).$$

- (c) Since $\{g_\alpha\}_{\alpha \in I}$ is a chain, such an α exists for every X , and $h \in P$ is an upper bound of the chain.

6. **Maximal element in P is a choice function:**

- (a) By Zorn's Lemma, there exists a maximal element $f \in P$.
- (b) Suppose $\text{dom}(f) \neq \mathcal{A}$. Then there exists some $X \in \mathcal{A}$ such that $X \notin \text{dom}(f)$.
- (c) Construct an extension of f by defining a new function f' :

$$\text{dom}(f') = \text{dom}(f) \cup \{X\},$$

$$f'(Y) = \begin{cases} f(Y) & \text{if } Y \in \text{dom}(f), \\ x & \text{for some } x \in X \text{ if } Y = X. \end{cases}$$

- (d) This contradicts the maximality of f , implying that $\text{dom}(f) = \mathcal{A}$.

7. Therefore, f is a choice function on $\mathcal{A}$, satisfying $f(X) \in X$ for all $X \in \mathcal{A}$.

$\therefore$ Zorn's Lemma implies the Axiom of Choice.

□

9 Conclusion

In this document, we have demonstrated the equivalence of several fundamental principles in the realm of set theory and mathematical logic. Starting from the Axiom of Choice (AC), we went through a sequence of important mathematical statements and showed how each one implies the next. The principles examined include:

1. Axiom of Choice (AC): The assertion that for any set of non-empty sets, a choice function exists that selects an element from each set.
2. Zermelo's Well-Ordering Theorem (WOT): The statement that every set can be well-ordered.
3. Transfinite Recursion (TR): A principle stating that one can define functions across well-ordered sets based on preceding values.
4. Hausdorff's Maximal Principle (HMP): The principle stating that every chain in a partially ordered set can be extended to a maximal chain.
5. Order-Extension Principle (OEP): The assertion that any partial order can be extended to a total order.
6. Existence of Basis for Vector Spaces (VB): The statement that every vector space has a basis.
7. Tychonoff's Theorem (TT): A topological theorem ensuring the compactness of the product of any collection of compact topological spaces.
8. Zorn's Lemma (ZL): The principle that every partially ordered set, where every chain has an upper bound, contains a maximal element.

Each of these principles has profound implications in various branches of mathematics. By establishing the equivalence among them, we can see the deep connections that these seemingly different concepts share.

The circular proof path taken in this document is summarized as follows:

$AC \implies WOT.$

$WOT \implies TR.$

$TR \implies HMP.$

$HMP \implies OEP.$

$OEP \implies VB.$

$VB \implies TT.$

$TT \implies ZL.$

$ZL \implies AC.$

This equivalence highlights the robustness of these principles. Accepting any one of them allows you to prove the others, demonstrating their foundational role in mathematics. By exploring this chain of implications, we gain insight into the nature of mathematical reasoning and the important position occupied by the Axiom of Choice.